Discrete Mathematics 310 (2010) 3429–3431

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Graphs with no M-alternating paths between two vertices: An update

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ARTICLE INFO

Article history:
Received 22 May 2009
Received in revised form 30 July 2010
Accepted 3 August 2010
Available online 21 August 2010

Keywords:
Necessary and sufficient condition
Perfect matching
n-extendable graph

ABSTRACT

In this paper, we obtain necessary and sufficient conditions for a graph G not to have an M-alternating path between two vertices in G.

All graphs in this paper are finite, undirected, connected and simple. For terminology and notation not defined in this paper, the reader is referred to [2].

Let G be a graph with a perfect matching M. Then G is said to be *n-extendable* if, for any matching M' of size n, M' is contained in a perfect matching of G. Let G be a connected graph. A vertex subset $S \subseteq V(G)$ is a *cutset* of G if $G - S$ is not connected. Let M be a perfect matching of G. A path P is an *M-alternating path* if the edges on P in turn appear in M and in $E(G) \setminus M$ alternately. In this paper, we shall consider all types of M-alternating paths which start with an edge in M (or $E(G) \setminus M$) and end with an edge in M (or $E(G) \setminus M$).

In the study of n-extendable graphs (see [4]), Aldred et al. [1] succeeded to use M-alternating paths to characterize n-extendable bipartite graphs. Later, Lou and Rao [3] began to characterize graphs in which there is no M-alternating path between two given vertices. For results on n-extendable graphs; see the survey [5].

In this paper, we complete characterization of graphs in which there are no M-alternating paths of each type between two vertices.

First, we give a known result.

Theorem 1 ([3]). *If x and y are vertices and M is a perfect matching in a graph G, then there is no M-alternating x, y-path starting and ending with edges in $E(G) \setminus M$ if and only if either*
(1) there is a cutset S in $V(G)$ such that $o(G - S) = |S|$ and x and y lie in distinct odd components of $G - S$, and x and y have distinct neighbours in S via edges of M; or
(2) there is a cutset S in $V(G)$ such that $o(G - S) = |S|$ and $xy \in M$ is a cut edge of an even component C of $G - S$ with $C - xy$ consisting of two odd components.

Theorem 1 gives a characterization of graphs in which there is no M-alternating path starting and ending with non-matching edges between two vertices. Next, we give another result of Lou and Rao [3].

Theorem 2. *Let G be a graph with a perfect matching M. If x and y are vertices in G such that $xy \notin M$, then there is no M-alternating x, y-path starting and ending with edges in M if and only if there is a cutset S in $V(G)$ such that $o(G-S) = |S|$, $x, y \in S$, $xx', yy' \in M$ and there are two odd components containing x' and y', respectively.*

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doi:10.1016/j.disc.2010.08.001

In Theorem 2, we have $xy \notin M$. If $xy \in M$, then, obviously, there is an M-alternating x, y-path starting and ending with the edge in M.

Now we give a theorem to characterize graphs in which there is no M-alternating path starting from x with a non-matching edge and ending at y with a matching edge.

Corollary 3. *Let G be a graph with a perfect matching M. If x and y are vertices in G such that $xy \notin M$, then there is no M-alternating x, y-path starting with an edge in $E(G) \setminus M$ and ending with an edge in M if and only if there is a cutset S in $V(G)$ such that $o(G - S) = |S|$, $xx', yy' \in M$, $x', y \in S$ and there are two odd components of $G - S$ containing x and y' respectively.*

Proof. Since $xy \notin M$, we have $xx', yy' \in M$ such that $x' \neq y$ and $y' \neq x$. There is no M-alternating path starting from x with an edge in $E(G) \setminus M$ and ending at y with an edge in M if and only if there is no M-alternating x', y-path starting and ending with edges in M. By Theorem 2, this corollary follows. □

Corollary 3 requires that $xy \notin M$. If $xy \in M$, then xy is not an M-alternating path from x to y starting with an edge in $E(G) \setminus M$. And there cannot be another M-alternating path from x to y starting with an edge in $E(G) \setminus M$ and ending with an edge in M since the only edge in M incident with y is xy.

Remark 1. Note that, by symmetry, Corollary 3 can also be restated with the roles of x and y exchanged.

Now we give the main theorem of this paper which characterizes graphs in which there is no M-alternating x, y-path of any type.

Theorem 4. *Let G be a graph with a perfect matching M. If x and y are vertices in G such that $xy \notin E(G)$, then there is no M-alternating x, y-path of any type if and only if there is a cutset S in $V(G)$ such that $o(G - S) = |S|$ and x and y lie in distinct even components of $G - S$.*

Proof. We prove sufficiency first. Since $o(G - S) = |S|$, the vertices of S are paired with vertices in the $|S|$ distinct odd components of $G - S$ by edges of M. Let C_1 and C_2 be two even components of $G - S$ such that $x \in V(C_1)$ and $y \in V(C_2)$.

Since even components of $G - S$ are covered by M, a path leaving C_1 must leave it on an edge not in M and follow such an edge to S. Any M-alternating path reaching S on an edge not in M next follows an edge of M to an odd component of $G - S$. The path can only leave that component on an edge not in M, and the argument repeats. Hence the path can never reach another even component of $G - S$.

Now we prove necessity. Let $xx', yy' \in M$ and $M' = M - \{xx', yy'\}$. Let G' be the graph obtained from G by contracting xx' and yy' into vertices v_x and v_y, respectively. Since G does not have an M-alternating x, y-path of any type, G' does not have an M'-alternating v_x, v_y-path. Since v_x and v_y are the only unsaturated vertices with respect to M', it follows that M' is a maximum matching. This implies that G' has no perfect matching, and $o(G' - S) > |S|$ for some $S \subset V(G')$. Since G has a perfect matching, $|V(G)|$ is even and hence $o(G' - S) \geq |S| + 2$. Moreover, $G' + v_xv_y$ has a perfect matching $M' \cup \{v_xv_y\}$. Thus, we have $o((G' + v_xv_y) - S) \leq |S|$. This is possible only if $o(G' - S) = |S| + 2$ and v_x and v_y lie in different odd components of $G' - S$, say C_x and C_y, respectively. Then $o(G - S) = |S|$, and x and y lie in $G[(V(C_x) \setminus \{v_x\}) \cup \{x, x'\}]$ and $G[(V(C_y) \setminus \{v_y\}) \cup \{y, y'\}]$, both of which are even components of $G - S$. □

Remark 2. Actually, the problems discussed in this paper and the previous one [3] are all equivalent to problems on maximum matchings in reduced graphs. Let G be a graph with a perfect matching M, and let x and y be distinct vertices in G. Let $\{xx', yy'\} \subset M$ and $M' = M - \{xx', yy'\}$. If $xy \in M$, we consider $xx' = yy' = xy$ and $M' = M - \{xy\}$.

- If $xy \notin E(G)$, then G has an M-alternating x, y-path if and only if M' is not a maximum matching of G', where G' is the graph obtained from G by contracting xx' and yy'.
- If $xy \notin E(G)$, then G has an M-alternating x, y-path starting and ending with edges of $E(G) - M$ if and only if M' is not a maximum matching in $G - \{x', y'\}$.
- If $xy \in M$, then G has an M-alternating x, y-path starting and ending with edges of $E(G) - M$ if and only if M' is not a maximum matching in $G - xy$.
- If $xy \notin M$, then G has an M-alternating x, y-path starting and ending with edges in M if and only if M' is not a maximum matching in $G - \{x, y\}$.
- If $xy \notin E(G)$, then G has an M-alternating x, y-path starting with an edge of $E(G) - M$ and ending with an edge of M if and only if M' is not a maximum matching in $G - \{x', y\}$.

Acknowledgements

The author would like to thank the referees for their good suggestions for shortening the proof of Theorem 4 and for revision of the paper.

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